

Shellability and Sphericity of the quasi-arc complex of the Möbius strip.

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Abstract

Shellability of a simplicial complex has many useful structural implications. In particular, it was shown in [4] by Danaraj and Klee that every shellable pseudo-manifold is a PL-sphere. The purpose of this paper is to prove the shellability of the quasi-arc complex of the Möbius strip. Along the way we provide elementary proofs of the shellability of the arc complex of the n -gon and the cylinder. In turn, applying the result of Danaraj and Klee, we obtain the sphericity of all of these complexes.

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1 Introduction

The arc complex $Arc(S)$ of a marked orientable surface S was introduced and studied by Harer [11] whilst investigating the homology of mapping class

groups of orientable surfaces. In [7], [8], Fomin, Shapiro and Thurston found there is a strong relation between cluster algebras and these arc complexes. They showed that $Arc(S)$ is a subcomplex of the cluster complex $\Delta(S)$ associated to the cluster algebra arising from S . Moreover, it was shown by Fomin and Zelevinsky, almost at the birth of cluster algebras, that the cluster complex of a cluster algebra has a polytopal realisation when the complex is finite, see [10]. Since $Arc(S)$ and $\Delta(S)$ coincide when S is an unpunctured surface, as a specific case, the well known fact that $Arc(n\text{-gon})$ is polytopal was rediscovered. Namely, it is dual to the associahedron.

In [5] Dupont and Palesi consider the quasi-arc complex of unpunctured non-orientable surfaces. Imitating the approach in [8] they describe how the ‘lengths’ of quasi-arcs are related. In doing so they discover what the analogue of a cluster algebra arising from non-orientable surfaces should be. A natural question is to ask what kind of structure the quasi-arc complex has in this setting. Here, in some sense, the marked Möbius strip M_n plays the role of the $n\text{-gon}$ - being the only non-orientable surface yielding a finite quasi-arc complex.

For $n \in \{1, 2, 3\}$ it is easy to check that the quasi-arc complex $Arc(M_n)$ of the Möbius strip is a polytope, see Figure 2. However, in general, due to the absence of a root system it is difficult to find a polytopal realisation.

It is shown in [2] that shellability of a simplicial complex is a necessary condition for it being polytopal. This paper is concerned with proving the following theorem.

Main Theorem. (Theorem 4.37). *$Arc(M_n)$ is shellable for $n \geq 1$.*

Shellability has its roots in polytopal theory where it turned out to be the missing piece of the puzzle for obtaining the Euler-Poincare formula. It has subsequently become a well established idea in combinatorial topology and geometry having some useful implications. For instance, Danaraj and Klee showed in [4] that every shellable pseudo-manifold is a PL-sphere. As a consequence, we obtain the following result.

Corollary. (Corollary 4.38). *$Arc(M_n)$ is a PL $(n - 1)$ -sphere for $n \geq 1$.*

The paper is organised as follows. In Section 2 we recall the work of Dupont and Palesi in [5]. Here we define the quasi-arc complex of a non-orientable surface and discuss why it is a pseudo-manifold, and when it is finite.

In Section 3 we firstly define shellability and recall some fundamental results. Next we restrict our attention to the $n\text{-gon}$ and to $C_{n,0}$ - the cylinder

with n marked points on one boundary component, and none on the other. In the interest of introducing key ideas of the paper early on, we present a short proof that both $\text{Arc}(n\text{-gon})$ and $\text{Arc}(C_{n,0})$ are shellable. As a consequence, applying the result of Danaraj and Klee, we rediscover the classical fact of Harer [12] that $\text{Arc}(n\text{-gon})$ and $\text{Arc}(C_{n,0})$ are PL-spheres.

Section 4 is dedicated to proving the shellability of $\text{Arc}(M_n)$ and occupies the bulk of the paper.

Acknowledgements

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2 Quasi-cluster algebras

This section recalls the work of Dupont and Palesi in [5].

Let S be a compact 2-dimensional manifold with boundary ∂S . Fix a set M of marked points in ∂S . The tuple (S, M) is called a bordered surface. We wish to exclude the cases where (S, M) does not admit any triangulation. As such, we do not allow (S, M) to be a monogon, digon or triangle.

Definition 2.1. An *arc* is a simple curve in (S, M) connecting two (not necessarily distinct) marked points.

Definition 2.2. A closed curve in S is said to be *two-sided* if it admits a regular neighbourhood which is orientable. Otherwise, it is said to be *one-sided*.

Definition 2.3. A *quasi-arc* is either an arc or a simple one-sided closed curve in the interior of S . Let $A^\otimes(S, M)$ denote the set of quasi-arcs in (S, M) considered up to isotopy.

It is well known that a closed non-orientable surface is homeomorphic to the connected sum of k projective planes $\mathbb{R}P^2$. Such a surface is said to have (non-orientable) genus k . Recall that the projective plane is homeomorphic to a hemisphere where antipodal points on the boundary are identified. A *cross-cap* is a cylinder where antipodal points on one of the boundary components are identified. We represent a cross cap as shown in Figure 1.

Hence, a closed non-orientable surface of genus k is homeomorphic to a sphere where k open disks are removed, and have been replaced with cross-caps. More generally, a compact non-orientable surface of genus k , with boundary, is homeomorphic to a sphere where more than k open disks are removed, and k of those open disks have been replaced with cross-caps.

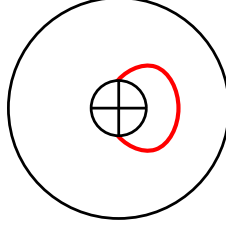


Figure 1: A picture of a crosscap together with a one-sided closed curve.

Definition 2.4. Two elements in $A^\otimes(S, M)$ are called **compatible** if there exists representatives in their respective isotopy classes that do not intersect in the interior of S .

Definition 2.5. A **quasi-triangulation** of (S, M) is a maximal collection of mutually compatible arcs in $A^\otimes(S, M)$. A quasi-triangulation is called a **triangulation** if it consists only of arcs, i.e there are no one-sided closed curves.

Proposition 2.6 ([5], Prop. 2.4.). *Let T be a quasi-triangulation of (S, M) . Then for any $\gamma \in T$ there exists a unique $\gamma' \in A^\otimes(S, M)$ such that $\gamma \neq \gamma'$ and $\mu_\gamma(T) := T \setminus \{\gamma\} \cup \{\gamma'\}$ is a quasi-triangulation of (S, M) .*

Definition 2.7. $\mu_\gamma(T)$ is called the **quasi-mutation** of T in the direction γ , and γ' is called the **flip** of γ with respect to T .

The flip graph of a bordered surface (S, M) is the graph with vertices corresponding to (quasi) triangulations and edges corresponding to flips. It is well known that the flip graph of triangulations of (S, M) is connected. Moreover, it can be seen that every one-sided closed curve, in a quasi-triangulation T , is bounded by an arc enclosing a Möbius strip with one marked point on the boundary. Therefore, if we perform a quasi-flip at each one-sided closed curve in T we arrive at a triangulation. As such, we get the following proposition.

Proposition 2.8 ([5], Prop. 2.12.). *The flip graph of quasi-triangulations of (S, M) is connected.*

Corollary 2.9. *The number of quasi-arcs in a triangulation of (S, M) is an invariant of (S, M) .*

Definition 2.10. The **quasi-arc complex** $Arc(S, M)$ is the simplicial complex on the ground set $A^\otimes(S, M)$ such that k -simplices correspond to sets of k mutually compatible quasi-arcs. In particular, the vertices in $Arc(S, M)$ are the elements of $A^\otimes(S, M)$ and the maximum simplices are the quasi-triangulations.

Together, Corollary 2.9 and Proposition 2.6 prove the following proposition.

Proposition 2.11. *$A^\otimes(S, M)$ is a pseudo-manifold. That is, each maximal simplex in $A^\otimes(S, M)$ has the same cardinality, and each simplex of co-dimension 1 is contained in precisely two maximal simplices.*

Theorem 2.12 ([5], Theorem 7.2). *Given a non-orientable bordered surface (S, M) then $Arc(S, M)$ is finite if and only if (S, M) is M_n , the Möbius strip with n marked points on the boundary.*

Moreover, $Arc(M_n)$ has some seemingly nice properties. Figure 2 shows that for $n \in \{1, 2, 3\}$ it is polytopal.

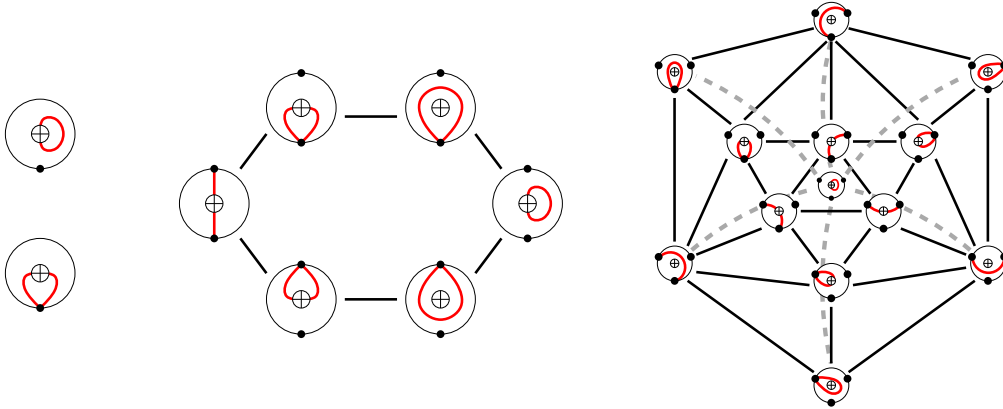


Figure 2: The quasi-arc complexes of M_1, M_2 and M_3 .

3 Shellability

In this section we recall some basic facts about shellability, and introduce the fundamental ideas used throughout this paper.

3.1 Definition of shellability and basic facts.

Definition 3.1. An n -dimensional simplicial complex is called **pure** if its maximal simplices all have dimension n .

Definition 3.2. Let Δ be a finite (or countably infinite) simplicial complex. An ordering C_1, C_2, \dots of the maximal simplices of Δ is a **shelling** if the complex $B_k := (\bigcup_{i=1}^{k-1} C_i) \cap C_k$ is pure and $(\dim(C_k) - 1)$ -dimensional for all $k \geq 2$.

Definition 3.3. The **simplicial join** $\Delta_1 * \Delta_2$ of two simplicial complexes Δ_1 and Δ_2 on disjoint ground sets has its faces being sets of the form $\sigma_1 \cup \sigma_2$ where $\sigma_1 \in \Delta_1$ and $\sigma_2 \in \Delta_2$.

The following proposition is a simple and well known result. For instance, see [1].

Proposition 3.4. *The simplicial join $\Delta_1 * \Delta_2$ is shellable if and only if the simplicial complexes Δ_1, Δ_2 are both shellable.*

In particular, Proposition 3.4 tells us that the cone over a shellable complex is itself shellable.

Proposition 3.5. *If $\Delta = \text{Arc}(S, M)$ then finding a shelling for Δ is equivalent to ordering the set of triangulations T_i of (S, M) so that $\forall k$ and $\forall j < k \exists i < k$ such that T_i is related to T_k by a mutation and $T_j \cap T_k \subseteq T_i \cap T_k$.*

Proof. Note that triangulations T_i of S correspond to maximal simplices in $\text{Arc}(S, M)$ and that partial triangulations $T_i \cap T_j$ correspond to simplices of $\text{Arc}(S)$. Note that $T_i \cap T_k$ is a $(\dim(T_k) - 1)$ -simplex iff T_k is a mutation away from T_i . Furthermore, since $B_k := (\bigcup_{i=1}^{k-1} T_i) \cap T_k$ must be pure and $(\dim(T_k) - 1)$ -dimensional for all $k \geq 2$, it follows that B_k is the union of $(\dim(T_k) - 1)$ -simplices. So we must have that $\forall j < k \exists i < k$ such that T_i is a mutation away from T_k and the partial triangulation $T_j \cap T_k$ is a face of $T_i \cap T_k$ (i.e $T_j \cap T_k \subseteq T_i \cap T_k$). □

Proposition 3.5 motivates Definition 3.6.

Definition 3.6. Given a subcollection of triangulations Γ of a surface S call Γ **shellable** if it admits an ordering of Γ such that $\forall k$ and $\forall j < k \exists i < k$ such that T_i is related to T_k by a mutation and $T_j \cap T_k \subseteq T_i \cap T_k$.

Definition 3.7. We say two sets of triangulations A, B are **equivalent** if their induced simplicial complexes are isomorphic, up to taking cones. If A and B are equivalent we write $A \equiv B$.

Remark 1. Let Δ_A denote the induced simplicial complex of a set of triangulations A . Note that taking a cone over Δ_A can be thought of as disjointly adding one particular arc to every triangulation in A .

The following proposition is just a special case of Proposition 3.4.

Proposition 3.8. *If $A \equiv B$ then A is shellable if and only if B is shellable.*

Notation:

- $\text{list}_{i=1}^n x_i$ is the ordering x_1, x_2, \dots, x_n of the set $\{x_i | 1 \leq i \leq n\}$.
- $\text{list}_{i \in I} x_i$ is any ordering of the set $\{x_i | i \in I\}$.
- Let $C_{n,0}$ denote the cylinder with n marked points on one boundary component and no marked points on the other. Fix an orientation on the boundary component containing marked points and cyclically label them $1, \dots, n$. Let $[i, j]$ denote the boundary segment $i \rightarrow j$. Note that $C_{n,0}$ arises as the partial triangulation of M_n consisting of a one-sided closed curve. We choose the canonical way of defining arcs on $C_{n,0}$.
- Let γ be an arc of $C_{n,0}$ with endpoints i, j . If γ encloses a cylinder with boundary $[j, i] \cup \gamma$ then $\gamma := \langle i, j \rangle$. If γ encloses a cylinder with boundary $[i, j] \cup \gamma$ then $\gamma := \langle j, i \rangle$, see Figure 3.

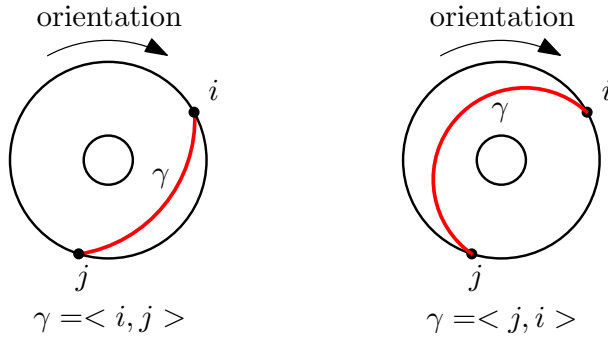


Figure 3: Notation for an arc γ of $C_{n,0}$

The following theorem provides a very useful application of shellability.

Theorem 3.9 (Danaraj and Klee, [4]). *Let Δ be a simplicial complex of dimension n . If Δ is a shellable pseudomanifold without boundary, then it is a PL n -sphere.*

3.2 Shellability of $\text{Arc}(C_{n,0})$.

The following proposition will help to prove the shellability of $\text{Arc}(M_n)$, and is introduced now to cement key ideas.

Proposition 3.10. *$\text{Arc}(C_{n,0})$ is shellable for $n \geq 1$.*

Proof. Consider the collection of triangulations $T(C_{n,0}^1) \subseteq T(C_{n,0})$ containing a loop at vertex 1. Note that by cutting along the loop we get the $(n+1)$ -gon (and a copy of $C_{1,0}$) for $n \geq 2$. We will prove by induction on n that $T(C_{n,0}^1)$ is shellable. For $n = 1$ the set $T(C_{1,0}^1) = T(C_{1,0})$ is trivially shellable. For $n = 2$ if we cut along the loop we get the triangle and $C_{1,0}$ which are both trivially shellable, so indeed $T(C_{2,0}^1)$ is shellable by Proposition 3.4.

Let $\text{Block}(i)$ be the set consisting of all triangulations in $T(C_{n,0}^1)$ containing the triangle with vertices $(1, 1, i)$ for some $i \in [2, n]$, see Figure 4.

Note that $\text{Block}(i) \equiv \prod_{j=1}^2 T(C_{m_j,0}^1)$ for $m_j < n$. By induction on n , $\text{Block}(i)$ is therefore the product of shellable sets. Taking the product of sets of triangulations corresponds to taking the join of their induced simplicial complexes. So Proposition 3.4 tells us that $\text{Block}(i)$ is shellable. Denote this shelling by $S(\text{Block}(i))$.

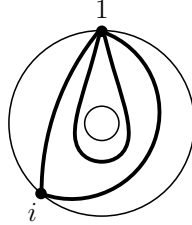


Figure 4: $\text{Block}(i)$ consists of all triangulations of this partial triangulation.

Claim 1. *The ordering $S(C_{n,0}^1) := \text{list}_{i=n}^2 S(\text{Block}(i))$ is a shelling for $T(C_{n,0}^1)$.*

$$\boxed{S(\text{Block}(n))}, \boxed{S(\text{Block}(n-1))}, \dots, \boxed{S(\text{Block}(2))}$$

Proof of Claim 1. Let S precede T in the ordering $S(C_{n,0}^1)$. Then $T \in \text{Block}(k)$ and $S \in \text{Block}(j)$ for $j \geq k$. Since $S(\text{Block}(k))$ is a shelling for $\text{Block}(k)$ (by inductive assumption) then we may assume $j > k$. The arc $\gamma = \langle k, 1 \rangle \in T$ is not compatible with the arc $\langle 1, j \rangle \in S$ so $\gamma \notin S$. Hence $T \cap S \subseteq T \cap \mu_\gamma(T)$. By Proposition 3.5 all that remains to show is that $\mu_\gamma(T)$ occurs before T in the ordering.

Note that we will have a triangle in T with vertices $(1, k, x)$ where $x \in [n, k + 1]$. And so $\mu_\gamma(T) \in \text{Block}(x)$. Since $x > k$, $\mu_\gamma(T)$ does precede T in the ordering. See Figure 5. Hence $T(C_{n,0}^1)$ is shellable.

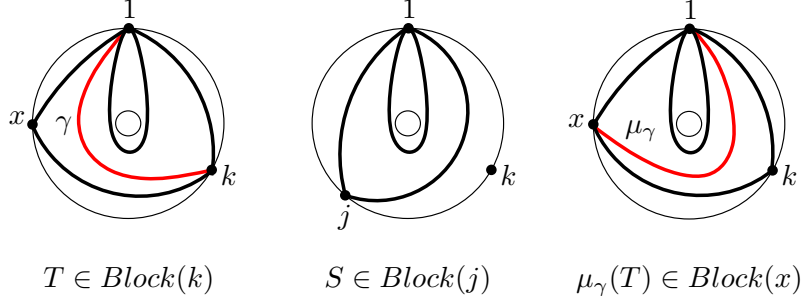


Figure 5

End of proof of Claim 1.

Similarly we can shell $T(C_{n,0}^i)$ in the same way $\forall i \in [1, n]$. Denote this shelling by $S(C_{n,0}^i)$

Claim 2. $S(\text{Arc}(C_{n,0})) := \text{list}_{i=1}^n S(C_{n,0}^i)$ is a shelling for $\text{Arc}(C_{n,0})$

Proof of Claim 2. Let S precede T in the ordering $S(\text{Arc}(C_{n,0}))$. Then $T \in S(C_{n,0}^k)$ and $S \in S(C_{n,0}^j)$ for $1 \leq j \leq k$. Since $S(C_{n,0}^k)$ is a shelling we may assume $j < k$. There will be a triangle in T with vertices (k, k, x) for some $x \in [1, n] \setminus \{k\}$.

If $x \in [j, k - 1]$ then mutate the loop at k to give $T' \in S(C_{n,0}^x)$. T' occurs before T in the ordering because $x \in [j, k - 1]$. Moreover since the loop at k cannot occur in S then $T \cap S \subseteq T \cap T'$. See Figure 6.

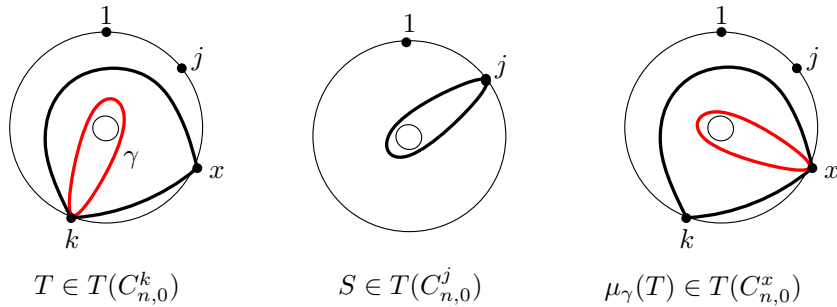


Figure 6: Case when $x \in [j, k - 1]$

If $x \in [k+1, j-1]$ then the arc $\gamma = \langle x, k \rangle$ in T is not compatible with the loop at j in S . So $T \cap S \subseteq T \cap \mu_\gamma(T)$. Moreover the way we constructed the shelling $S(C_{n,0}^k)$ in Claim 1 means that $\mu_\gamma(T)$ precedes T in the ordering. See Figure 7.

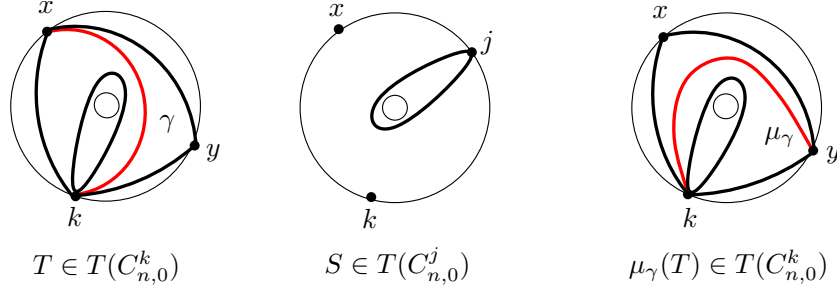


Figure 7: Case when $x \in [k+1, j-1]$

End of proof of Claim 2.

□

Corollary 3.11. *Arc(n -gon) is shellable for $n \geq 3$*

Proof. Follows immediately from Claim 1.

□

Applying Theorem 3.9 we rediscover the classical result of Harer, [12].

Corollary 3.12. *Arc($C_{n,0}$) and Arc(n -gon) are PL-spheres of dimension $n-2$ and $n-4$, respectively.*

4 Main Theorem

In Section 3 we achieved shellability of a complex by grouping facets into blocks and finding a ‘shelling order’ in terms of these blocks. The task was then simplified to finding a shelling of the blocks themselves. Here we essentially follow the same strategy, twice. However, on the second iteration of the process we require a specific shelling of the blocks - in general an arbitrary shelling would not suffice.

Definition 4.1. Let $T(M_n^\circ) \subseteq T(M_n)$ consist of all triangulations of M_n (i.e no quasi-triangulations containing a one-sided curve).

Definition 4.2. Let γ be an arc in $T \in T(M_n^\circ)$. Call γ a **crosscap arc** (c-arc) if $M_n \setminus \{\gamma\}$ is orientable. (Informally, a c-arc is an arc that necessarily passes through the crosscap). Let (i, j) denote a c-arc with endpoints i and j .

Definition 4.3. Call a triangulation $T \in T(M_n^\circ)$ a **crosscap triangulation** (c-triangulation) if every arc in T is a c-arc. Let $T(M_n^\otimes) \subseteq T(M_n^\circ)$ consist of all c-triangulations.

Definition 4.4. Let γ be an arc in $T \in T(M_n^\circ)$ that is not a c-arc. Call γ a **bounding arc** (b-arc) if it mutates to a c-arc.

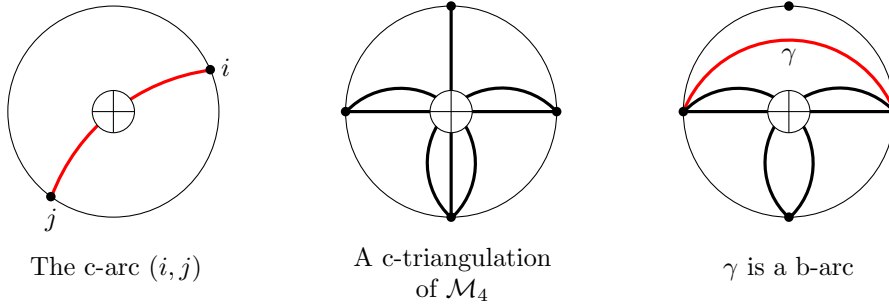


Figure 8

4.1 Reducing the problem to c-triangulations.

Lemma 4.5. If $T(M_n^\otimes)$ is shellable then so is $T(M_n^\circ)$.

Proof. Consider $I := \{i_1, \dots, i_k\} \subseteq [1, n]$. Let $\Gamma_I^{(k)}$ consist of all triangulations $T \in T(M_n^\circ)$ such that there is a c-arc in T with endpoint j if and only if $j \in I$. Note that this condition implies the existence of an arc or boundary segment $\langle i_m, i_{m+1} \rangle$ (where $i_{k+1} := i_1$) in every triangulation $T \in \Gamma_I^{(k)}$ $\forall m \in [1, k]$.

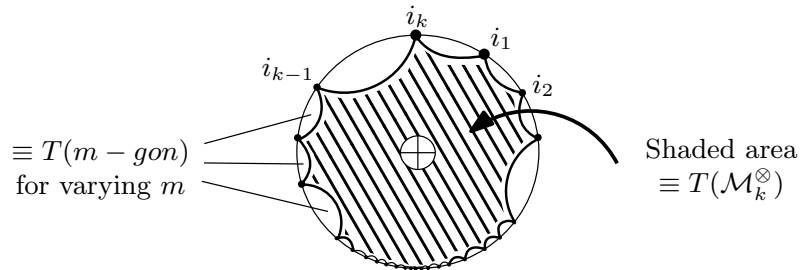


Figure 9: $\Gamma_I^{(k)}$

By assumption $T(M_n^\otimes)$ is shellable, and by Corollary 3.11 $T(m\text{-gon})$ is also shellable. Hence $\Gamma_I^{(k)}$ is the product of shellable collections of triangulations, and so is shellable by Proposition 3.4. Denote this shelling by $S(\Gamma_I^{(k)})$.

Claim 3. Let $Block(k) := \text{list}_{I \in [1,n]^{(k)}} S(\Gamma_I^{(k)})$. Then $\text{list}_{k=n}^1 Block(k)$ is a shelling for $T(M_n^\circ)$.

Proof of Claim 3. Let S precede T in the ordering. Then $S \in Block(j)$ and $T \in Block(k)$ where $j \geq k$. In particular, $T \in S(\Gamma_{I_1}^{(k)})$ and $S \in S(\Gamma_{I_2}^{(j)})$ for some $I_1, I_2 \in \mathcal{P}([1,n])$ where $|I_1| \leq |I_2|$. Since $S(\Gamma_I^{(k)})$ is a shelling we may assume $I_1 \neq I_2$.

Suppose that every b-arc in T is also an arc in S . Then $I_2 \subseteq I_1$, and since $|I_1| \leq |I_2|$ this implies $I_1 = I_2$. So we may assume there is at least one b-arc $\gamma \in T$ that is not an arc in S . Since $\gamma \notin S$, $T \cap S \subseteq T \cap \mu_\gamma(T)$. Moreover, since γ is a b-arc, $\mu_\gamma(T) \in Block(k+1)$. Hence $\mu_\gamma(T)$ precedes T in the ordering, see Figure 10.

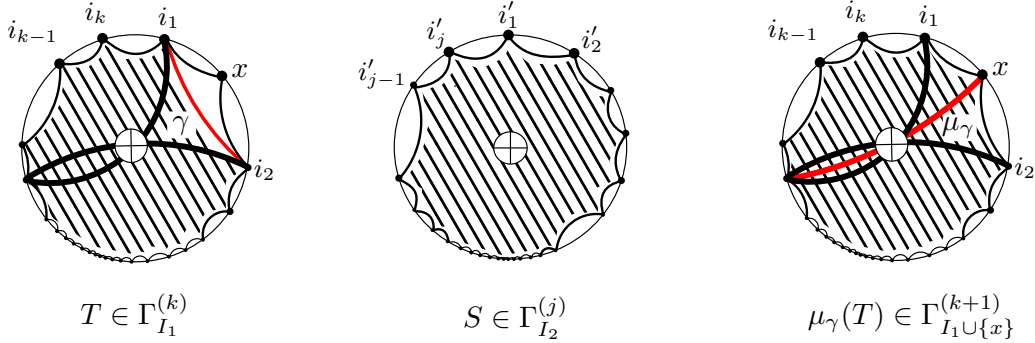


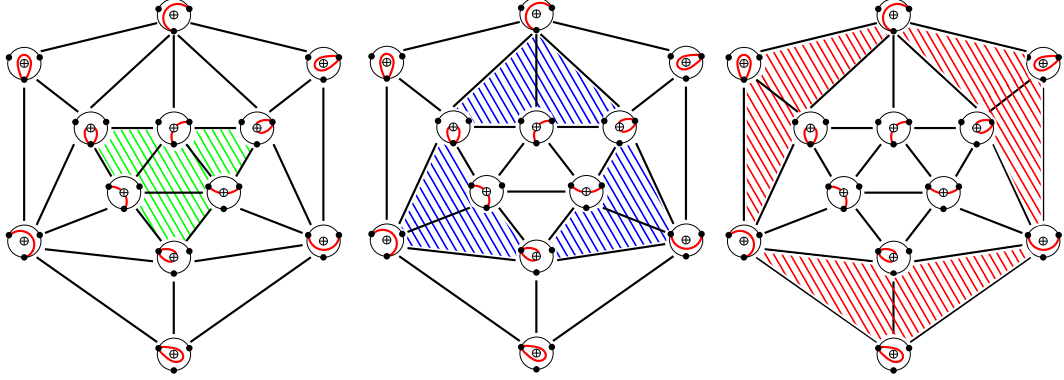
Figure 10

End of proof of Claim 3.

□

The idea behind Lemma 4.5 is that we are decomposing $T(M_n^\circ)$ into blocks, and ordering these blocks. The ordering is chosen in such a way that if we manage to individually shell the blocks themselves, we'll have a shelling of $T(M_n^\circ)$. Figure 11 shows the block structure of $T(M_3^\circ)$.

In particular, we realise that to shell a block it is sufficient to find a shelling of $T(M_n^\otimes)$. We will split this into two cases: n even and n odd.

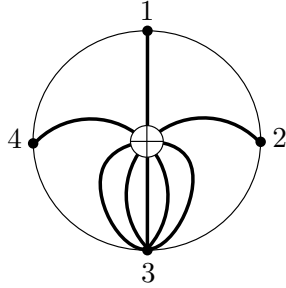


$$Block(3) = \Gamma_{\{1,2,3\}}^3 \quad Block(2) = \Gamma_{\{1,2\}}^2 \cup \Gamma_{\{2,3\}}^2 \cup \Gamma_{\{1,3\}}^2 \quad Block(1) = \Gamma_{\{1\}}^1 \cup \Gamma_{\{2\}}^1 \cup \Gamma_{\{3\}}^1$$

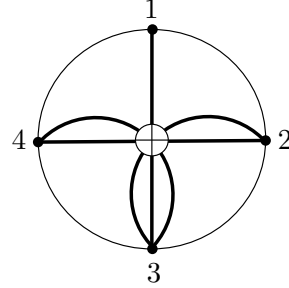
Figure 11: Block structure of $T(M_3^o)$

4.2 Shellability of $T(M_n^\otimes)$ for even n .

Let $D_{\{(1, \frac{n}{2}+1)\}}^n$ consist of all triangulations of $T(M_n^\otimes)$ containing the c-arc $(1, \frac{n}{2}+1)$ but containing no other c-arcs $(i, \frac{n}{2}+i) \forall i \in [2, n]$. See Figure 12.



Example of a triangulation
in $D_{\{(1,3)\}}^4$



Example of a triangulation
not in $D_{\{(1,3)\}}^4$

Figure 12

Definition 4.6. Let $T \in D_{\{(1, \frac{n}{2}+1)\}}^n$ and γ a c-arc in T . $\gamma = (i, j)$ for some $i \in [1, 1 + \frac{n}{2}]$ and $j \in [1 + \frac{n}{2}, 1]$. Define the **length** of γ as follows:

- If $i = j = 1$, $l(\gamma) := n + 1$.
- Otherwise, $l(\gamma) := |[i, j]|$.

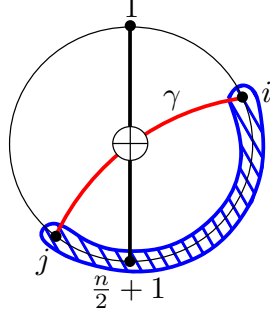


Figure 13: If $i \neq 1$ or $j \neq 1$ then the number of marked points in the shaded tube equals $l(\gamma)$.

Definition 4.7. Let \mathcal{X}_1^n be the partial triangulation of M_n consisting of the c-arcs $(1, \frac{n}{2} + 1)$, $(2, \frac{n}{2} + 1)$, $(n, \frac{n}{2} + 1)$. Additionally, let $T(\mathcal{X}_1^n)$ denote the triangulations in $D_{\{(1, \frac{n}{2} + 1)\}}^n$ containing the c-arcs $(1, \frac{n}{2} + 1)$, $(2, \frac{n}{2} + 1)$, $(n, \frac{n}{2} + 1)$. Similarly, let \mathcal{X}_2^n be the partial triangulation of M_n consisting of the c-arcs $(1, \frac{n}{2} + 1)$, $(1, \frac{n}{2})$, $(n, \frac{n}{2} + 2)$. Let $T(\mathcal{X}_2^n)$ denote the triangulations in $D_{\{(1, \frac{n}{2} + 1)\}}^n$ containing the c-arcs $(1, \frac{n}{2} + 1)$, $(2, \frac{n}{2} + 1)$, $(n, \frac{n}{2} + 1)$. See Figure 14.

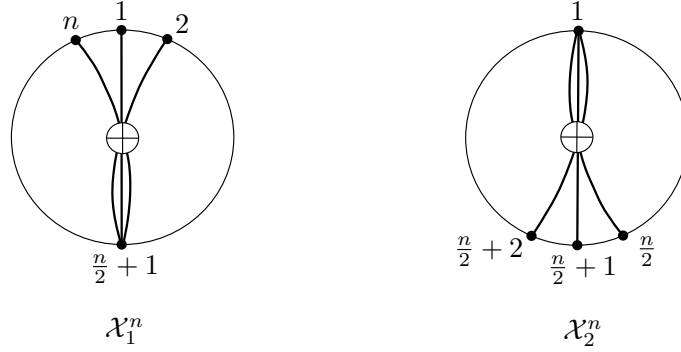


Figure 14

Lemma 4.8. $D_{\{(1, \frac{n}{2} + 1)\}}^n = T(\mathcal{X}_1^n) \sqcup T(\mathcal{X}_2^n)$. Moreover, for any c-arc $\gamma \neq (1, \frac{n}{2} + 1)$ in T we have the following:

- $l(\gamma) \leq \frac{n}{2}$ if $T \in T(\mathcal{X}_1^n)$.
- $l(\gamma) \geq \frac{n}{2} + 2$ if $T \in T(\mathcal{X}_2^n)$.

Proof. A triangulation T in $D_{\{(1, \frac{n}{2} + 1)\}}^n$ will contain either the c-arc $(2, \frac{n}{2} + 1)$ or the c-arc $(1, \frac{n}{2} + 2)$.

Assume the c-arc $(2, \frac{n}{2} + 1)$ is in T . We will show, by induction on i , the c-arc of maximal length in T with endpoint $i \in [2, \frac{n}{2} + 1]$ must be the c-arc (i, x) where $x \in [\frac{n}{2} + 1, \frac{n}{2} + i - 1]$.

Let γ be the c-arc in T of maximal length with endpoint 2. Let j be the other endpoint of γ and suppose for a contradiction $j \in [\frac{n}{2} + 2, n]$. Since $(2, \frac{n}{2} + 1) \in T$ then, as T is a c-triangulation, $(2, x) \in T \forall x \in [\frac{n}{2} + 1, j]$. In particular $\beta := (2, \frac{n}{2} + 2) \in T$ - which contradicts $T \in D_{\{(1, \frac{n}{2} + 1)\}}^n$. See Figure 15.

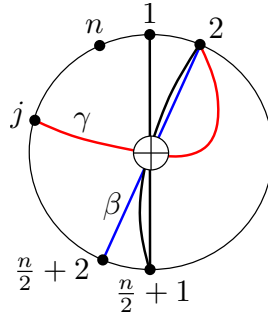


Figure 15

By induction, the c-arc α of maximal length in T with endpoint $i - 1$ is the c-arc $(i - 1, x)$ where $x \in [\frac{n}{2} + 1, \frac{n}{2} + i - 2]$. Let γ be the c-arc in T of maximal length with endpoint i . Let j be the other endpoint of γ and suppose $j \in [\frac{n}{2} + i, n]$. But by the maximality of α there will be a c-arc $(i, y) \forall y \in [x, j]$. In particular there will be a c-arc $\beta := (i, \frac{n}{2} + i)$. See Figure 16.

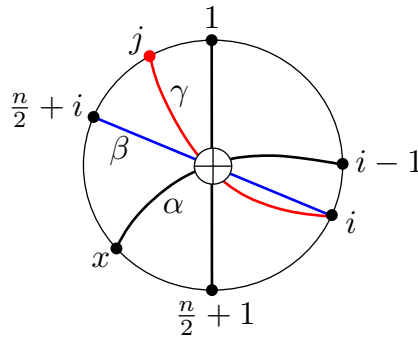


Figure 16

If we supposed $(1, \frac{n}{2} + 2)$ was an arc in T , then an analogous argument shows that $T \in T(\mathcal{X}_2)$.

□

Corollary 4.9. *Let $S \in T(\mathcal{X}_1^n)$ and $T \in T(\mathcal{X}_2^n)$ then $S \cap T = \{(1, \frac{n}{2} + 1)\}$*

Proof. It follows from the fact that, excluding the c-arc $(1, \frac{n}{2} + 1)$, the maximal length of any c-arc in \mathcal{X}_1^n is less than or equal to $\frac{n}{2}$, and the minimal length of any c-arc in \mathcal{X}_2^n is greater than or equal to $\frac{n}{2} + 2$.

□

Corollary 4.10. *The triangulation T_{max} in Figure 17 is the unique triangulation in $T(\mathcal{X}_1^n)$ such that $\sum_{\gamma \in T_{max}} l(\gamma)$ is maximal. The triangulation T_{min} is the unique triangulation in $T(\mathcal{X}_2^n)$ such that $\sum_{\gamma \in T_{min}} l(\gamma)$ is minimal. More explicitly,*

$$T_{max} := \{(1, \frac{n}{2} + 1)\} \cup \{(i, \frac{n}{2} + i - 1) | i \in [2, \frac{n}{2} + 1]\} \cup \{(i, \frac{n}{2} + i - 2) | i \in [3, \frac{n}{2} + 1]\}.$$

$$T_{min} := \{(1, \frac{n}{2} + 1)\} \cup \{(i, \frac{n}{2} + i + 1) | i \in [1, \frac{n}{2}]\} \cup \{(i, \frac{n}{2} + i + 2) | i \in [1, \frac{n}{2} - 1]\}.$$

Proof. Consider the partial triangulation \mathcal{P} of \mathcal{X}_1^n consisting of all the c-arcs of maximal length. Namely the c-arcs $(i, \frac{n}{2} + i - 1) \forall i \in [2, \frac{n}{2} + 1]$. \mathcal{P} cuts M_n into (2 triangles and) quadrilaterals bounded by the two boundary segments $[i, i + 1]$, $[\frac{n}{2} + i - 1, \frac{n}{2} + i]$ and the two c-arcs $(i, \frac{n}{2} + i - 1)$, $(i + 1, \frac{n}{2} + i) \forall i \in [3, \frac{n}{2}]$. Let T be a triangulation of \mathcal{P} such that $T \in T(\mathcal{X}_1^n)$. Notice that $(i, \frac{n}{2} + i) \notin T$ by definition of $D_{\{(1, \frac{n}{2} + 1)\}}^n$, hence $(i + 1, \frac{n}{2} + i - 1) \in T \forall i \in [3, \frac{n}{2} + 1]$ and so $T = T_{max}$. Moreover, since $l(i + 1, \frac{n}{2} + i - 1) = l(i, \frac{n}{2} + i - 1) - 1$ then T is the unique triangulation in $T(\mathcal{X}_1^n)$ such that $\sum_{\gamma \in T} l(\gamma)$ is maximal.

Analogously we get the result regarding unique minimality of T_{min} .

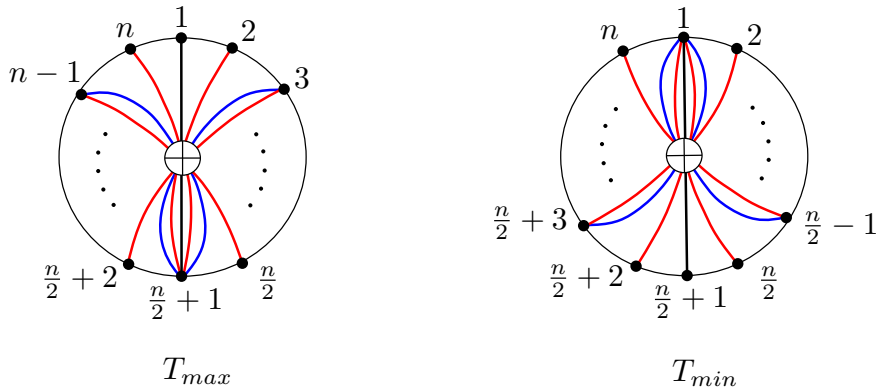


Figure 17

□

Definition 4.11. Call a c-arc $(i, \frac{n}{2} + i)$ of M_n a **diagonal** arc.

Definition 4.12. Consider a c-arc γ in a triangulation of \mathcal{X}_1^n . If $l(\gamma) = \frac{n}{2}$ then call γ a **max** arc.

Definition 4.13. Consider a c-arc γ in a triangulation of \mathcal{X}_2^n . If $l(\gamma) = \frac{n}{2} + 2$ then call γ a **min** arc.

Consider a partial triangulation of \mathcal{X}_1^n containing two max arcs. Cutting along these max arcs we will be left with two regions. Let R be the region that doesn't contain the diagonal arc $(1, \frac{n}{2} + 1)$. Note R will contain $2k$ marked points for some $k \in \{2, \dots, \frac{n}{2}\}$.

Lemma 4.14. *The set of triangulations of R such that no max arcs occur in R is equivalent to $T(\mathcal{X}_1^{2(k-1)})$.*

Proof. Collapse the quadrilateral $(1, 2, \frac{n}{2} + 1, n)$ to a c-arc and relabel marked points as shown in Figure 18.

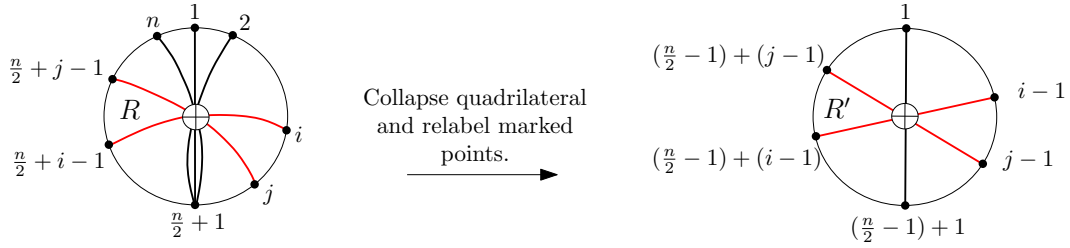


Figure 18

Max arcs in R correspond to diagonal arcs in R' . Furthermore, up to a relabelling of vertices, triangulating R' so that no diagonal arcs occur in the triangulation is precisely triangulating $\mathcal{X}_1^{2(k-1)}$ so that no diagonal arcs occur.

□

Remark 2. Using induction we realise that Lemma 4.14 tells us that $D_{\{(1, \frac{n}{2}+1)\}}^n$ has the same flip structure as the set of all Dyck paths of length $n - 2$. In particular, triangulations in $D_{\{(1, \frac{n}{2}+1)\}}^n$ correspond to Dyck paths, and arcs appearing in those triangulations correspond to nodes in the Dyck lattice. This correspondence is indicated in Figure 19 and is best viewed in colour.

Definition 4.15. Let $i \in \{1, 2\}$. Call an arc γ in $T \in T(\mathcal{X}_i^n)$ **\mathcal{X} -mutable** if $\mu_\gamma(T) \in T(\mathcal{X}_i^n)$.

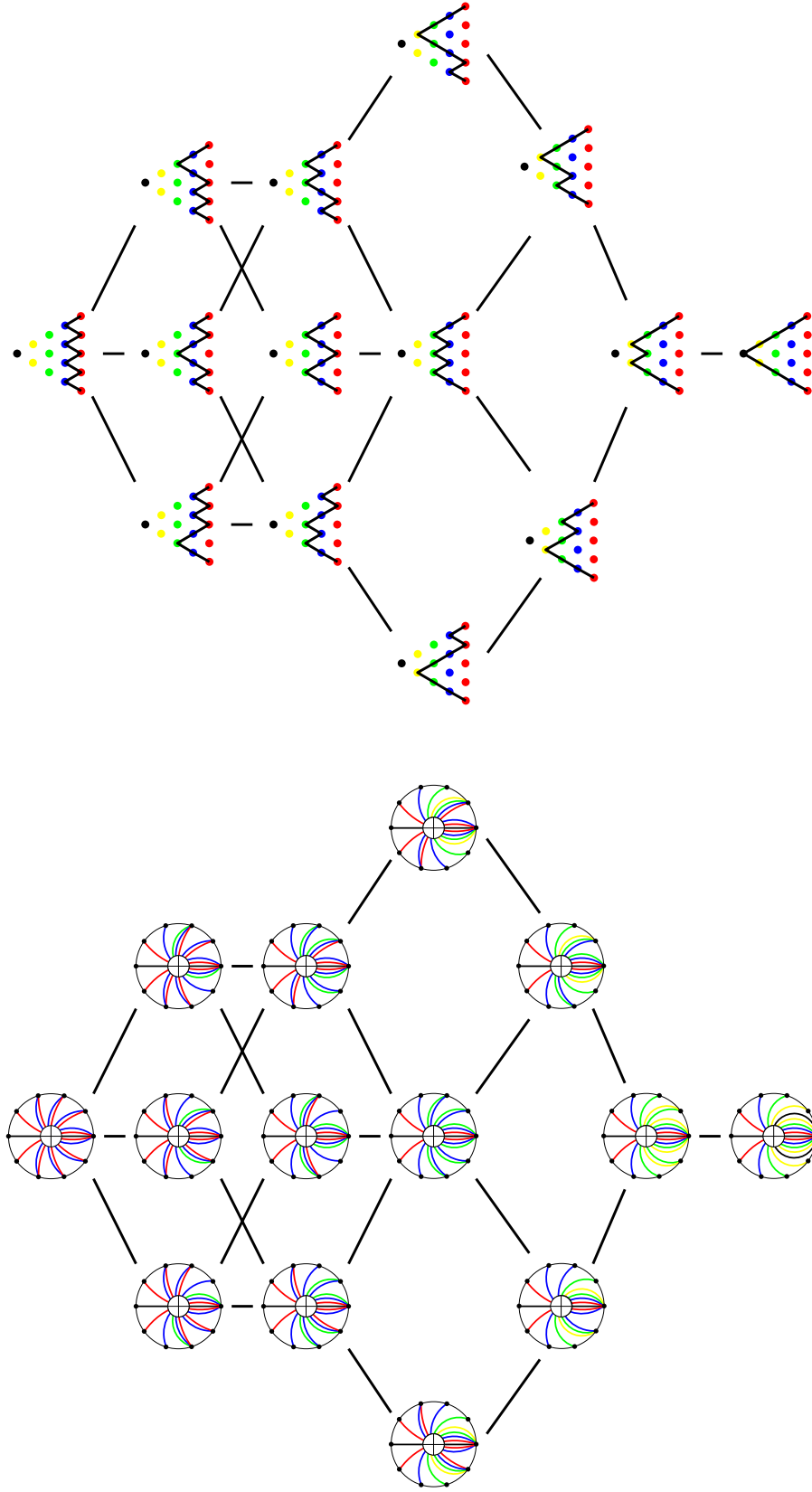


Figure 19: $D_{\{(1, \frac{n}{2}+1)\}}^n$ and Dyck paths. See Remark 2 for an explanation of their connection.

Definition 4.16. Let γ be an \mathcal{X} -mutable arc in a triangulation $T \in D_{\{(1, \frac{n}{2}+1)\}}^n$, and let γ' be the arc γ mutates to. Call γ **upper-mutable** if $l(\gamma') > l(\gamma)$ and **lower-mutable** if $l(\gamma') < l(\gamma)$.

Definition 4.17. Call a shelling \mathcal{S} of $T(\mathcal{X}_1^n)$ ($T(\mathcal{X}_2^n)$) an **upper (lower)** shelling if for any triangulation $T \in \mathcal{S}$ and any upper (lower) mutable arc γ in T , $\mu_\gamma(T)$ precedes T in the ordering.

Definition 4.18. Let \mathcal{I} be the set of all max arcs of $D_{\{(1, \frac{n}{2}+1)\}}^n$, excluding the max arcs $\alpha_1 := (1, \frac{n}{2} + 1), \alpha_2 := (\frac{n}{2} + 1, n)$.

Lemma 4.19. If $T \in T(\mathcal{X}_1^n)$ doesn't contain a max arc $m \in \mathcal{I}$ then there exists an upper mutable arc γ strictly contained between the endpoints of m , see Figure 20.

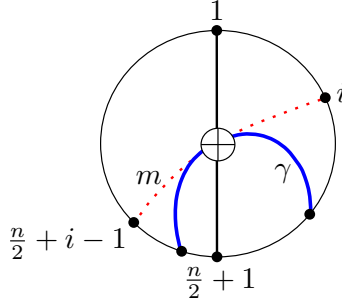


Figure 20

Proof. If $n \in \{2, 4\}$ then $\mathcal{I} = \emptyset$ and there is nothing to prove. So assume $n \geq 6$.

Suppose $m = (i, \frac{n}{2} + i - 1) \in \mathcal{I}$ is not in the triangulation T . We will show there exists a c-arc strictly contained between the endpoints of m .

Let (i, x) be the c-arc of maximum length in T connected to i . Since $m \neq (i, x)$ then $x \in [\frac{n}{2} + 1, \frac{n}{2} + i - 2]$. Moreover, by maximality of (i, x) , $(i + 1, x) \in T$. So indeed there is a c-arc in T strictly contained between the endpoints of m , see Figure 21.

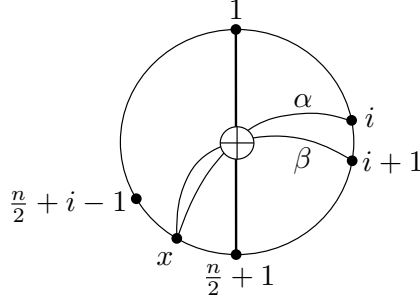


Figure 21

Of the c-arcs that are strictly contained between the endpoints of m , let $\gamma = (j_1, j_2)$ be an arc of minimum length. We will show that γ is upper mutable.

By minimality of γ the c-arc $(j_1, j_2 - 1)$ is not in T . Hence the c-arc $(j_1 - 1, j_2)$ must be in T . Likewise the c-arc $(j_1, j_2 + 1) \in T$. So γ is contained in the quadrilateral $(j_1, j_1 - 1, j_2, j_2 + 1)$. Hence mutating γ gives $\gamma' = (j_1 - 1, j_2 + 1)$. $l(\gamma) < l(\gamma')$ so γ is indeed upper mutable, see Figure 22.

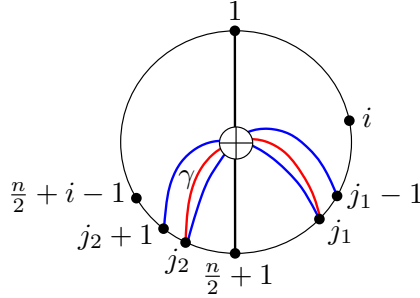


Figure 22

□

Lemma 4.20. *There exists an upper shelling for $T(\mathcal{X}_1^n)$. Denote this by $S(\mathcal{X}_1^n)$.*

Proof. Let $\Psi_{\{\gamma_1, \dots, \gamma_k\}}$ be the collection of triangulations in $T(\mathcal{X}_1^n)$ containing the max arcs $\gamma_1, \dots, \gamma_k, \alpha_1, \alpha_2$ and no other max arcs. By Lemma 4.14 we know that $\Psi_{\{\gamma_1, \dots, \gamma_k\}} \equiv \prod_{i=1}^j T(\mathcal{X}_1^{m_i})$.

Moreover, by induction on the trivial base case when $n = 2$, and using Proposition 3.4, we get that there is an upper shelling for $\Psi_{\{\gamma_1, \dots, \gamma_k\}}$. Denote this shelling by $S(\Psi_{\{\gamma_1, \dots, \gamma_k\}})$.

Claim 4. Let $\text{Block}(k) := \text{list}_{J \in \mathcal{I}^{(k)}} S(\Psi_J)$. Then $\text{list}_{k=\frac{n}{2}-2}^0 \text{Block}(k)$ is an upper shelling for $T(\mathcal{X}_1^n)$.

Proof of Claim 4. Let $T, S \in T(\mathcal{X}_1^n)$ and suppose S precedes T in the proposed ordering. Then $T \in \Psi_{J_1}$ and $S \in \Psi_{J_2}$ where $J_1, J_2 \in \mathcal{P}([1, n])$ and $|J_1| \leq |J_2|$. W.l.o.g. we may assume $J_1 \neq J_2$ since by induction $S(\Psi_{J_1})$ is an upper shelling.

As $|J_1| \leq |J_2|$ and $J_1 \neq J_2$ there is a max arc m in S that is not in T . By Lemma 4.19 there is an upper mutable arc γ in T strictly contained between the endpoints of m . Moreover γ and m are not compatible so $S \cap T \subseteq \mu_\gamma(T) \cap T$. And $\mu_\gamma(T)$ precedes T in the ordering because of the upper shelling $S(\Psi_{J_1})$.

End of proof of Claim 4.

□

An analogous argument proves the following lemma.

Lemma 4.21. *There exists a lower shelling for $T(\mathcal{X}_2^n)$. Denote this by $S(\mathcal{X}_2^n)$.*

Definition 4.22. Call a c-arc γ in a triangulation $T \in D_{\{(1, \frac{n}{2}+1)\}}^n$ **special mutable** if any of the following is true:

- $T \in T(\mathcal{X}_1^n)$ and γ is upper mutable.
- $T \in T(\mathcal{X}_2^n)$ and γ is lower mutable.
- γ mutates to a diagonal c-arc.

Lemma 4.23. *For any $T \in T(\mathcal{X}_1^n) \setminus \{T_{\max}\}$, T_{\max} is connected to T by a sequence of lower mutations.*

Proof. By Lemma 4.19 we can keep performing mutations on upper mutable arcs until we reach a triangulation containing every max arc. By Corollary 4.10 the only triangulation in $T(\mathcal{X}_1^n)$ that contains every max arc is T_{\max} . Hence T is connected to T_{\max} by a sequence of upper mutations. Equivalently, T_{\max} is connected to T by a sequence of lower mutations.

□

Lemma 4.24. *Let $T \in D_{\{(1, \frac{n}{2}+1)\}}^n$ and let P_T be the partial triangulation of M_n consisting of all the special mutable arcs in T . Then any triangulation of P_T cannot contain the diagonal c-arc $(i, \frac{n}{2} + i) \forall i \in \{2, \dots, \frac{n}{2}\}$.*

Proof. Assume $T \in T(\mathcal{X}_1^n)$. An analogous argument works if $T \in T(\mathcal{X}_2^n)$. We prove the lemma via induction on the upper shelling order of $T(\mathcal{X}_1^n)$.

The first triangulation in the upper shelling ordering is T_{max} . The special mutable arcs in T_{max} are $(i, \frac{n}{2} + i - 2) \forall i \in [3, \frac{n}{2} + 1]$. However, the c-arc $(i, \frac{n}{2} + i - 2)$ is not compatible with the diagonal c-arc $(i - 1, \frac{n}{2} + i - 1)$. And so ranging i over $3, \dots, \frac{n}{2} + 1$ proves the base inductive case.

Let γ be a lower mutable arc in a triangulation $T \in T(\mathcal{X}_1^n)$. By Lemma 4.23, to prove the lemma it suffices to show that the special mutable arcs in $\mu_\gamma(T)$ prevent the same diagonal c-arcs as the special mutable arcs in T . Let β_1, β_2 be the c-arcs containing γ in a quadrilateral. See Figure 23.

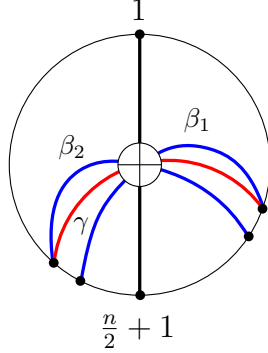


Figure 23

The arcs β_1 and β_2 may be special mutable in T but in $\mu_\gamma(T)$ they definitely won't be. The implication of this is that β_1 and β_2 may be c-arcs in P_T , and prevent certain diagonal arcs, but $\beta_1, \beta_2 \notin P_{\mu_\gamma(T)}$ so μ_γ needs to make up this difference. Indeed, it does make up the difference as the diagonal arcs not compatible with either β_1 or β_2 are precisely the diagonal arcs not compatible with μ_γ .

□

Lemma 4.25. *In each c-triangulation T of M_n there is at least one diagonal arc.*

Proof. Let us assume, for a contradiction, that there is no diagonal arc in T . Without loss of generality, we may assume that the c-arc connected to 1, of maximum length, is $\gamma = (1, j_1)$ for some $j_1 \in [1, \frac{n}{2}]$. (Otherwise just flip the picture.)

Let $\gamma_2 = (2, j_2)$ be the c-arc of maximum length in T that is connected to 2. If $j_2 > \frac{n}{2}$ then by maximality of γ_1 there is a c-arc $(2, \frac{n}{2})$. Hence,

$j_2 \in [j_1, \frac{n}{2} + 1]$. Inductive reasoning shows that the c-arc connected to $j_1 - 1$ in T , of maximum length, is $\gamma_{j-1} = (j - 1, x)$ for some $x \in [j, \frac{n}{2} + j_1 - 2]$. However, then by the maximality of γ_{j-1} we must have $(j_1, \frac{n}{2} + j_1) \in T$. This gives a contradiction, and so the lemma is proved. \square

Lemma 4.26. $T(M_n^\otimes)$ is shellable for even n .

Proof. Let \mathcal{K} be the collection of diagonal c-arcs of M_n . Consider $I = \{\gamma_1, \dots, \gamma_k\} \subseteq \mathcal{K}$ and let D_I^n consist of all triangulations of $T(M_n^\otimes)$ containing every diagonal c-arc in I , and no diagonal c-arcs in $\mathcal{K} \setminus I$. The set of c-triangulations $T(R)$ of a region R cut out by two diagonal c-arcs, so that no other diagonal c-arcs occur in the region, is equivalent to $D_{\{(1, \frac{m}{2} + 1)\}}^m$ for some $m \in [2, n - 2]$. See Figure 24.

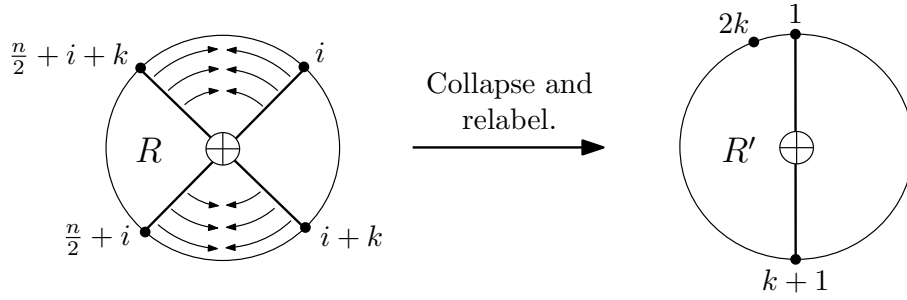


Figure 24: $T(R) \equiv T(R') = D_{\{(1, k+1)\}}^{2k}$

Choose $\text{list}_{i=1}^2 S(\mathcal{X}_i^m)$ to be the ordering of $D_{\{(1, \frac{m}{2} + 1)\}}^m$. Take the disjoint union of these orderings, over all the regions cut out by diagonal c-arcs in I , to get an ordering of D_I^n . Denote this ordering by $O(D_I^n)$.

Claim 5. $\text{list}_{k=\frac{n}{2}}^1 \text{Block}(k)$ is a shelling for $T(M_n^\otimes)$.

Where $\text{Block}(k) := \text{list}_{I \in \mathcal{K}^{(k)}} O(D_I^n)$.

Proof of Claim 5. Let $T, S \in T(M_n^\otimes)$ and suppose S precedes T in the ordering. Then $T \in O(D_{I_1}^n)$ and $S \in O(D_{I_2}^n)$ for some $I_1, I_2 \in \mathcal{P}(\mathcal{K})$ where $|I_1| \leq |I_2|$.

If there is a region R in T that contains a special mutable arc γ , such that γ is not an arc in S , then $\mu_\gamma(T)$ precedes T in the ordering and $S \cap T \subseteq \mu_\gamma(T) \cap T$.

So suppose that for every region R of T all special mutable arcs in that region are also arcs in S . Then by Lemma 4.24 $I_2 \subseteq I_1$. Since $|I_1| \leq |I_2|$ we must have $I_1 = I_2$.

If $O(D_I^n)$ was a shelling for D_I^n then the proof would be finished. However, in general, it is not. To understand how we should proceed let us consider $D_{\{(1, \frac{n}{2}+1)\}}^n$.

By definition, $O(D_{\{(1, \frac{n}{2}+1)\}}^n) = \text{list}_{i=1}^2 S(\mathcal{X}_i^n)$. Let T be the first triangulation of $S(\mathcal{X}_2)$ and let $S \in S(\mathcal{X}_1)$. Corollary 4.9 tells us that the only arc T and S share in common is the diagonal c-arc $(1, \frac{n}{2} + 1)$. If $n = 2$ then $O(D_{\{(1,2)\}}^2) = S, T$ is a shelling for $D_{\{(1,2)\}}^2$. However, if $n \geq 4$ then there are at least 4 arcs in S and T . Hence, $\mu_\gamma(T) \notin S(\mathcal{X}_1^n)$ for any arc γ in T , since $\mu_\gamma(T)$ and S can share at most two arcs in common.

However, as $n \geq 4$ the first triangulation of $S(\mathcal{X}_2^n)$ contains (at least one) arc γ that mutates to a diagonal c-arc. And so $\mu_\gamma(T)$ contains more diagonal c-arcs than T . Hence $\mu_\gamma(T)$ precedes T in the overall ordering for $T(M_n^\otimes)$.

End of proof of Claim 5.

□

4.3 Shellability of $T(M_n^\otimes)$ for odd n .

In the even case diagonal arcs were a key ingredient in the shelling of $T(M_n^\otimes)$. We will see 'diagonal triangles' play the same role in the odd case. For the duration of this section we fix $n = 2k + 1$.

Definition 4.27. A triangle in M_n comprising of two c-arcs $(i, i + k)$, $(i, i + k + 1)$ and the boundary segment $(i + k, i + k + 1)$ for some $i \in [1, n]$ is called a **diagonal triangle** (d-triangle). Additionally, call i the **special vertex** of the d-triangle.

Definition 4.28. Let \mathcal{Y}^n be the partial triangulation of M_n containing the d-triangle $(k + 1, 2k + 1, 1)$. And let $T(\mathcal{Y}^n) \subseteq T(M_n^\otimes)$ consist of all c-triangulations of M_n containing the d-triangle $(k + 1, 2k + 1, 1)$, and no other d-triangles. See Figure 25.

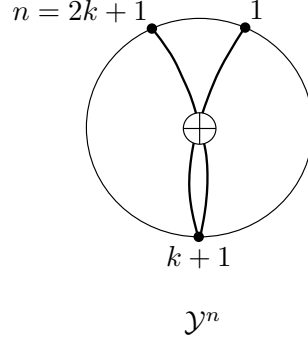


Figure 25

Definition 4.29. Let $T \in T(\mathcal{Y}^n)$ and γ a c-arc in T . $\gamma = (i, j)$ for some $i \in [1, k + 1]$ and $j \in [k + 1, n]$. Define the **length** of γ as $l(\gamma) := j - i + 1$, see Figure 26.

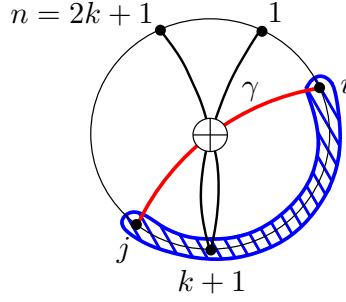


Figure 26: 'Number of marked points in shaded tube' = $l(\gamma)$.

Lemma 4.30. *The max length of any c-arc in $T \in \mathcal{Y}^n$ is $k + 1$.*

Proof. Given $T \in T(\mathcal{Y}^n)$ we will prove by induction on $i \in [1, k + 1]$ that there is no c-arc in T , with endpoint $k + i$, of length greater than $k + 1$. For $i = 1$ this trivially holds. Now assume the statement is true for i . Then there is a c-arc $\gamma = (x, k + i)$ in T where $x \in [i, k + 1]$. But the c-arc of maximum length, with endpoint $k + i + 1$, that is compatible with γ is $\beta = (x, k + i + 1)$. If $x \in [i + 1, k + 1]$ then indeed $l(\beta) \leq k + 1$. If $x = i$ then we have a d-triangle $(i, k + i, k + i + 1)$ with special vertex i - which is forbidden. So indeed $l(\beta) \leq k + 1$. \square

Lemma 4.31. $T(\mathcal{Y}^n) \equiv T(\mathcal{X}_1^{n+1})$. As such, $T(\mathcal{X}_1^{n+1})$ induces an upper shelling of $T(\mathcal{Y}^n)$. Denote this upper shelling by $S(\mathcal{Y}^n)$.

Proof. Add a marked point to the d-triangle $(k + 1, 2k + 1, 1)$ in \mathcal{Y}^n and relabel the marked points. Adding the c-arc $(1, k + 2)$ we get \mathcal{X}_1^{n+1} . Lemma

4.30 tells us the maximum length of an arc in $T \in T(\mathcal{Y}^n)$ is $k + 1$. And since the length of a max arc in $T(\mathcal{X}_1^{n+1})$ is also $k + 1$ then $T(\mathcal{Y}^n) \equiv T(\mathcal{X}_1^{n+1})$. See Figure 27. □

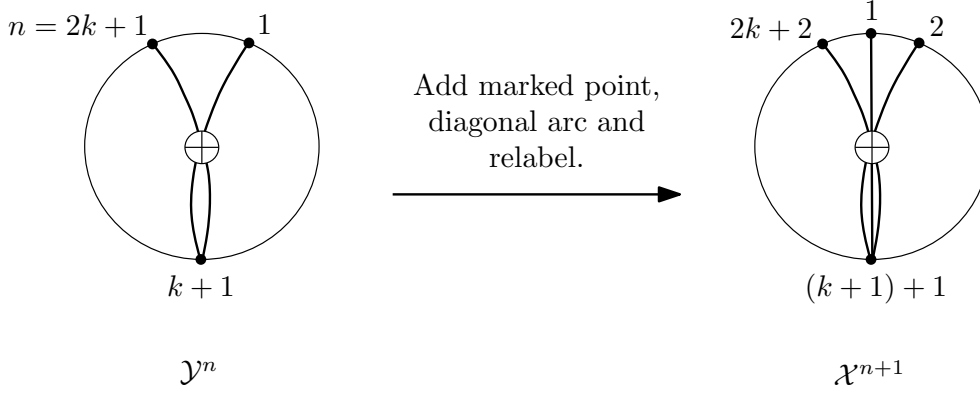


Figure 27

Lemma 4.32. *For any $T \in T(\mathcal{Y}^n)$ there are an odd number of d-triangles in T . Moreover, the collection of triangulations of any region cut out inbetween d-triangles, such that no other d-triangles occur, is equivalent to $T(\mathcal{Y}^m)$ for some $m < n$.*

Proof. We will show that if there are two d-triangles there must in fact be a third. Additionally we'll show the collection of (legitimate) triangulations in any region cut out inbetween the three d-triangles is equivalent to $T(\mathcal{Y}^m)$ for some $m < n$. And applying induction on this we will have proved the lemma.

Suppose there are at least two d-triangles in a c-triangulation T . Without loss of generality we may assume the two d-triangles $(k + 1, 2k + 1, 1)$ and $(i, i + k, i + k + 1)$ are in T , for some $i \in [1, k]$. See Figure 28.

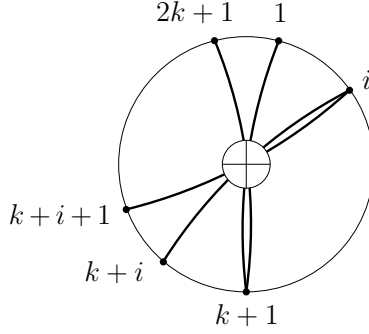


Figure 28

We will show there is a third d-triangle with special vertex $z \in [i+k+1, 2k+1]$. Note that if $(i+1, i+k+1) \in T$ then the d-triangle $(i+k+1, i, i+1) \in T$. Similarly, if $(k, 2k+1) \in T$ then the d-triangle $(2k+1, k+1, k) \in T$.

So suppose $(i+1, i+k+1), (k, 2k+1) \notin T$. This then implies $(i+1, x) \in T$ for some $x \in [i+k+2, 2k]$, and $(k, y) \in T$ for some $y \in [i+k+2, 2k]$. In turn, by induction, there is a d-triangle with special vertex $z \in [x, y]$. See Figure 29.

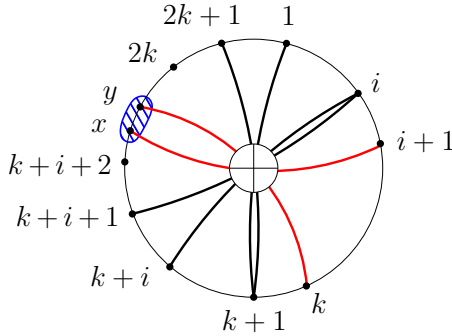


Figure 29: By induction there is a d-triangle with its special vertex in the shaded region.

What remains to prove is that each region cut out by these three d-triangles is equivalent to $T(\mathcal{Y}^m)$ for some $m < n$.

Consider the d-triangles $(k+1, 2k+1, 1)$ and $(i, i+k, i+k+1)$ with special vertices $k+1$ and i , respectively. Let R be the region bounded by the c-arcs $(1, k+1)$, $(i, i+k)$ and the boundary segments $[1, i], [k+1, k+i]$. Collapsing the boundary segment $[i, k+1]$ to a point and collapsing $[k+i, 1]$ to a boundary segment preserves the notion of length in R . After collapsing we see that triangulating R (so that no d-triangles occur) is equivalent to triangulating \mathcal{Y}^{2i-1} . See Figure 30.

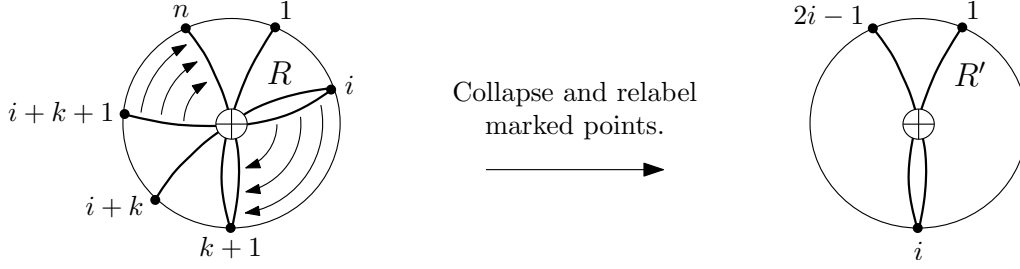


Figure 30

Similarly the collection of triangulations of either of the other two regions cut out by the three d-triangles is equivalent to $T(\mathcal{Y}^m)$ for some $m < n$. This completes the proof. \square

Definition 4.33. Let $T \in T(\mathcal{Y}^n)$ and let γ be a c-arc in T . Call γ **special mutable** if it is upper mutable or $\mu_\gamma(T)$ contains more d-triangles than T .

Lemma 4.34. Let $T \in T(\mathcal{Y}^n)$ and let P_T be the partial triangulation of M_n consisting of all special mutable arcs in T . Then for any triangulation of P_T there is no d-triangle with special vertex $i \forall i \in [1, \dots, n] \setminus k+1$.

Proof. We follow the same idea used in Lemma 4.24. Namely, we will prove the lemma by induction on the shelling order of $S(\mathcal{Y}_n)$.

Let T_1 be the first triangulation in the shelling. Note $\gamma_i = (i, k+i-1)$ is a special mutable c-arc in $T_1 \forall i \in [2, k+1]$. Moreover γ_i is not compatible with the c-arc $(i-1, k+i)$. Hence there is no d-triangle with special vertex $i-1$ or $k+i \forall i \in [2, k+1]$. This proves the base inductive case.

Let $T \in T(\mathcal{Y}^n)$. What remains to show is that for any lower mutable arc $\gamma \in T$, the d-triangles incompatible with P_T are precisely the d-triangles incompatible with $P_{\mu_\gamma(T)}$.

So let γ be a lower mutable arc in T . Let β_1, β_2 be the c-arcs of the quadrilateral containing γ . See Figure 31.

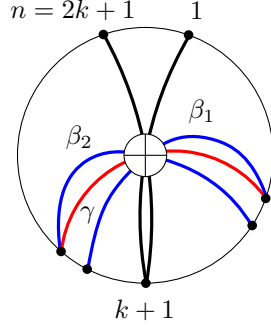


Figure 31

Note that β_1 and β_2 could be upper mutable in T , but they will definitely not be upper mutable in $\mu_\gamma(T)$. Analogous to the proof of Lemma 4.24, to prove the lemma it suffices to show μ_γ is incompatible with all the d-triangles incompatible with either β_1 or β_2 .

This follows from the fact that a c-arc $\alpha = (x, k+y)$ of length less than k is incompatible with d-triangles with special vertex $z \in [y, x-1] \cup [k+y+1, k+x]$. See Figure 32.

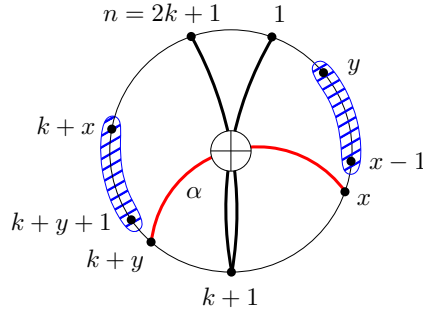


Figure 32: α is incompatible with d-triangles whose special vertex lies in one of the shaded regions.

□

An analogous argument to Lemma 4.25 proves the following lemma.

Lemma 4.35. *In each c-triangulation T of M_n there is at least one d-triangle.*

Lemma 4.36. *$T(M_n^\otimes)$ is shellable for odd n .*

Proof. Let \mathcal{K} be the collection of d-triangles of M_n that can occur in a triangulation without containing any other d-triangles. Consider $I = \{\Delta_1, \dots, \Delta_k\} \subseteq$

\mathcal{K} and let D_I^n consist of all triangulations of $T(M_n^\otimes)$ containing every d-triangle in I , and no d-triangles in $\mathcal{K} \setminus I$.

By Lemma 4.32, each region cut out inbetween the d-triangles in I is shellable. Taking the product of these shellings over all regions gives us a shelling for D_I^n . Denote this shelling by $S(D_I^n)$.

Claim 6. $\text{list}_{k=\frac{n}{2}}^1 \text{Block}(k)$ is a shelling for $T(M_n^\otimes)$.

Where $\text{Block}(k) := \text{list}_{I \in \mathcal{K}^{(k)}} S(D_I^n)$.

Proof of Claim 6. Let $T, S \in T(M_n^\otimes)$ and suppose S precedes T in the ordering. Then $T \in S(D_{I_1}^n)$ and $S \in S(D_{I_2}^n)$ for some $I_1, I_2 \in \mathcal{P}(\mathcal{K})$ where $|I_1| \leq |I_2|$.

If there is a region R in T that contains a special arc γ , such that γ is not an arc in S , then $\mu_\gamma(T)$ precedes T in the ordering and $S \cap T \subseteq \mu_\gamma(T) \cap T$.

So suppose that for every region R of T all special arcs in that region are also arcs in S . Then by Lemma 4.34 $I_2 \subseteq I_1$. Since $|I_1| \leq |I_2|$ we must have $I_1 = I_2$. And since $S(D_{I_1}^n)$ is a shelling for $D_{I_1}^n$ the claim is proved.

End of proof of Claim 6.

□

Together Lemma 4.26 and Lemma 4.36 prove $T(M_n^\otimes)$ is shellable for all $n \geq 1$.

Returning to our example of M_3 , Figure 33 shows a shelling of $T(M_3^\circ)$ that we can obtain through our construction.

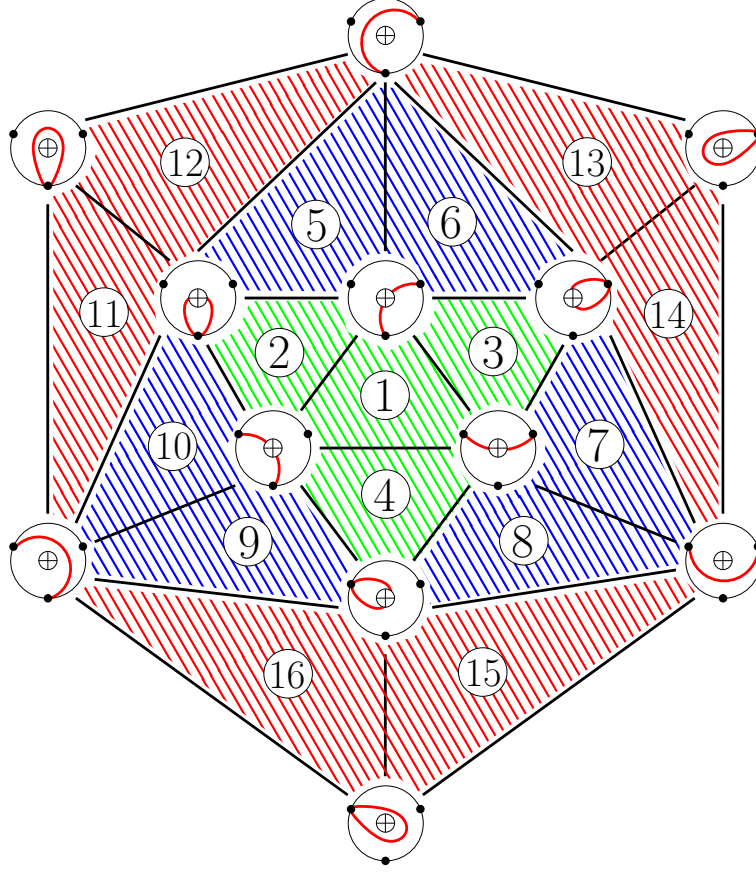


Figure 33: Shelling of $T(M_3^o)$

4.4 Proof of Main Theorem.

Theorem 4.37 (Main Theorem). *$\text{Arc}(M_n)$ is shellable for $n \geq 1$.*

Proof. Let \mathcal{C} consist of all quasi-triangulations of M_n containing the one-sided closed curve. Cutting along the one-sided curve in M_n we are left with the marked surface $C_{n,0}$. $\text{Arc}(C_{n,0})$ is shellable by Proposition 3.10. Since \mathcal{C} is the cone over $\text{Arc}(C_{n,0})$, then by Proposition 3.4 it is also shellable. Let $S(\mathcal{C})$ denote a shelling for \mathcal{C} . Lemma 4.5 together with Lemma 4.26 and Lemma 4.36 proves that $T(M_n^o)$ is shellable. Let $S(M_n^o)$ be a shelling of $T(M_n^o)$.

Claim 7. $S(M_n) := S(M_n^o), S(\mathcal{C})$ is a shelling for $T(M_n)$

Proof of Claim 7. Suppose $S, T \in S(M_n)$ and S precedes T in the ordering. Without loss of generality we may assume $S \in S(M_n^o)$ and $T \in S(\mathcal{C})$. Since T contains the one-sided closed curve γ , and $\gamma \notin S$ then $S \cap T \subseteq \mu_\gamma(T) \cap T$. Moreover, $\mu_\gamma(T) \in S(M_n^o)$ so precedes T in the ordering.

End of proof of Claim 7.

□

Corollary 4.38. *$\text{Arc}(M_n)$ is a PL $(n - 1)$ -sphere for $n \geq 1$.*

Proof. Follows immediately from Theorem 3.9 and Theorem 4.37.

□

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